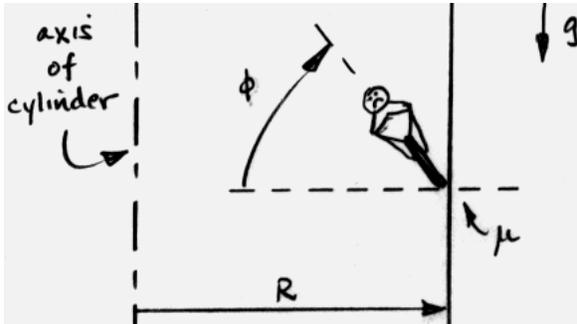


University of California, Berkeley
 Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 7

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1. We will need to use fictitious forces to solve this problem easily. Fictitious forces are never necessary, but they often simplify problems greatly.



(a.) We want to know the minimum speed the cyclist needs not to slip down the side. The force of static friction must be mg to hold him up, so we require that $\mu N = mg$. The force of static friction can be less than μN , but we are setting it to the maximum to see what the limit is. Thus we need $N > mg/\mu$. The only force acting horizontally in the system is the normal force, so it must entirely provide the centripetal acceleration, which is v^2/R . We thus obtain

$$\frac{mv^2}{R} \geq \frac{mg}{\mu} \Rightarrow v \geq \sqrt{\frac{gR}{\mu}}$$

(b.) We now need to consider the fictitious centrifugal force. The cyclist is in an accelerating frame of reference because he is moving in a circle. To correctly apply Newton's second law in the cyclist's frame, we must introduce the centrifugal force, which points outward with magnitude mv^2/R . In the frame of the cyclist there are four forces: the normal force and the centrifugal force cancel each other, and the friction and gravity cancel each other. This is required to insure that, by definition, the cyclist isn't accelerating in his own frame of reference. The problem is now reduced to a torque balance to find the angle at which the cyclist is stable.

We choose the point of contact as the origin. There are two torques, τ_g caused by gravity and τ_c caused by the centrifugal force. Both act at the center of mass. (This is important to note: fictitious forces always act at the center of mass!) We are assuming that the size of the cyclist l is very small in comparison with the radius R , so

$$\tau_g = +mgl \cos \phi \quad \tau_c = -\frac{mlv^2}{R} \sin \phi$$

We want to know the angle where the cyclist is about to slip, so the normal force is mg/μ , equal to the centripetal force mv^2/R . Substituting for mv^2/R in the above equation,

$$mgl \cos \phi = mgl \frac{1}{\mu} \sin \phi \Rightarrow \tan \phi = \mu$$

So when the cyclist is about to slip, he rides at an angle

$$\phi = \tan^{-1} \mu$$

(c.) Taking $\mu = 0.6$ and $R = 5$ meters, we find that the cyclist must ride at a speed of least 9.0 meters per second, or 29.5 ft/sec, or 20.1 mph. On a road bike this is a mellow cruising speed. At this minimum speed, the angle the cyclist must make with the horizontal is 31 degrees. We caution you not to try this at home; it's tough to get up to speed without crashing!

2. K&K problem 6.24

This problem is similar to many pulley problems that you have seen before. We need to apply both Newton's second law and the torque equation to solve it. We denote the (positive downward) acceleration of the falling mass as a . There are two forces on it in the vertical direction, tension and gravity. Newton's second law requires

$$Mg - T = Ma$$

We now need to apply the torque equation to both drums. For each drum we choose its own

center as the origin. Each drum feels only one torque, the torque from the tension. This has a magnitude $\tau = TR$ in both cases. Notice that both of these torques have the same sign, thus the drums will tend to angularly accelerate in the same direction. Writing the torque equation for each drum, with angular accelerations α_1 for the top drum and α_2 for the bottom drum,

$$TR = I\alpha_1 \quad TR = I\alpha_2$$

where the moments of inertia of both disks are the same, $I = MR^2/2$. From these equations it is easy to see that $\alpha_1 = \alpha_2 \equiv \alpha = TR/I$, so the angular accelerations are both

$$\alpha = \frac{2T}{MR}$$

We now need to find a relation between a and α . The linear acceleration due to each disk is given simply by $a = \alpha R$. There are two disks, both unwinding with the same angular acceleration α , so the linear acceleration of the bottom one is just $a = 2\alpha R$. The previous equation becomes

$$a = \frac{4T}{M} \Rightarrow T = \frac{Ma}{4}$$

Plugging this into the very first equation that we got from Newton's law, we find the initial acceleration of the drum, assuming that it moves straight down, to be

$$a = \frac{4}{5}g$$

Will the drum in fact move straight down? For the moment assuming that the answer is "yes", consider a (downward accelerating but nonrotating) frame with its origin at the (instantaneous) point of tangency between the lower drum and the tape. In this frame, the CM of the drum experiences a downward force mg and an upward fictitious force $\frac{4}{5}mg$ which does not quite compensate it. Therefore it feels a net downward force $\frac{1}{5}mg$. About the chosen origin this force causes a net (clockwise) torque, which causes the lower drum to swing to the left like a pendulum bob in this frame. This contradicts our

assumption of a pure downward motion. Therefore the actual motion will be more complicated than this problem asks you to assume.

3. K&K problem 6.27

We need to apply both Newton's law and the torque equation. The forces on the yo-yo horizontally are the force F and the friction f . The vertical forces are the normal force and gravity, which immediately tell us that $N = Mg$. We want to find the maximum force we can apply with the yo-yo not slipping. It is important to note that the force of friction, which stops the disk from slipping, is controlled by the coefficient μ_s of *static* friction because the surface of a rolling wheel is at rest with respect to the ground. Since we are concerned with the maximum allowed force, we will consider the maximum allowed friction, which is $\mu_s N$. Newton's law gives us

$$F - \mu_s Mg = Ma$$

The moment of inertia of the yo-yo is $I = MR^2/2$. Because we want the yo-yo to roll without slipping, we can use $a = \alpha R$. The torque equation gives us

$$\mu_s MgR - Fb = I\alpha = \frac{1}{2}MRa$$

We want to solve these two equations for F , the maximum allowed force. Eliminating a , we get

$$F - \mu_s Mg = 2\mu_s Mg - 2F\frac{b}{R}$$

Solving for F , we get

$$F = \mu_s Mg \frac{3R}{R + 2b}$$

Since $R > b$ the applied force F is always larger than the frictional force, so the yo-yo always accelerates to the right.

4. We will solve this problem symbolically and plug in numbers at the end. This is always a good practice because it makes it a lot easier to go back and check your work for correct dimensions and reasonable results for limiting cases.

(a.) Let the disk have mass M and radius R , and the two men each have mass m . If the men are momentarily at a radius r from the center of the disk, the total moment of inertia is given by

$$I(r) = \frac{1}{2}MR^2 + 2mr^2$$

The initial angular velocity of the disk is ω_0 , so, when $r = R$, the initial angular momentum of the system is

$$L = I\omega_0 = \left(\frac{1}{2}M + 2m\right)R^2\omega_0$$

There are no net torques acting on the system, so L is conserved. We can use L conservation, $I(r)\omega(r) = I(R)\omega_0$, to obtain the angular velocity of the system as a function of the radius of the men:

$$\omega(r) = \omega_0 \frac{MR^2 + 4mR^2}{MR^2 + 4mr^2}$$

The final angular velocity ω' is just the angular velocity when the men reach the center, $\omega(r = 0)$.

$$\omega' = \omega_0 \frac{M + 4m}{M} = \left(1 + \frac{4m}{M}\right)\omega_0$$

Plugging in values, we find that the final angular velocity is 1.5 revolutions per second.

The factor by which the kinetic energy has increased is

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}I(0)\omega'^2}{\frac{1}{2}I(R)\omega_0^2}$$

Evaluating this, we find

$$\frac{K_f}{K_i} = \frac{\frac{1}{4}MR^2\left(1 + \frac{4m}{M}\right)^2\omega_0^2}{\frac{1}{4}MR^2\omega_0^2 - mR^2\omega_0^2}$$

Simplifying,

$$\begin{aligned}\frac{K_f}{K_i} &= 1 + \frac{4m}{M} \\ \Delta K &= \frac{4m}{M}K_i\end{aligned}$$

For the masses in the problem, K_f/K_i is equal to 3. The rotational kinetic energy is tripled.

(b.) The extra kinetic energy comes from the work that the men must do against the (fictitious) centrifugal force to make their way from the edge of the turntable to the center. This is the qualitative statement which the problem requests. Optionally, one can perform a quantitative analysis:

Each man pushes against the centrifugal force to get to the center. The work they do is converted to rotational kinetic energy. The centrifugal force on each man is given by

$$F_c = m(\omega(r))^2r = m\omega_0^2r \left(\frac{1 + 4m/M}{1 + 4mr^2/(MR^2)}\right)^2$$

The work done is just F_c integrated from zero to R , doubled since each man does the same amount of work.

$$\Delta W = 2m\omega_0^2 \left(1 + \frac{4m}{M}\right)^2 \int_0^R \frac{r dr}{\left(1 + \frac{4mr^2}{MR^2}\right)^2}$$

You can look this up in a table, or notice that the top is proportional to the derivative of the bottom, so antidifferentiating is not too hard:

$$\Delta W = -2m\omega_0^2 \left(1 + \frac{4m}{M}\right)^2 \left(\frac{MR^2/8m}{\left(1 + \frac{4mr^2}{MR^2}\right)^2}\right)_0^R$$

Evaluating this, we get

$$\Delta W = \frac{MR^2\omega_0^2}{4} \left\{ \left(1 + \frac{4m}{M}\right)^2 - \left(1 + \frac{4m}{M}\right) \right\}$$

Multiplying this out,

$$\Delta W = \frac{MR^2\omega_0^2}{4} \left(\frac{4m}{M} + \frac{16m^2}{M^2}\right)$$

Simplifying,

$$\begin{aligned}\Delta W &= mR^2\omega_0^2 + \frac{4m^2}{M}R^2\omega_0^2 \\ &= \left(1 + \frac{4m}{M}\right)mR^2\omega_0^2\end{aligned}$$

This is $4m/M$ times the initial kinetic energy, so, as expected, ΔW is equal to the kinetic energy gain ΔK that we already calculated.

(c.) We want to find where the maximum centrifugal force is felt. This is just a maximization problem. Differentiate F_c with respect to r and set it to zero, and also check the endpoints.

$$\frac{dF_c}{dr} = \frac{d}{dr} \left(m\omega_0^2 r \frac{(1 + 4m/M)^2}{(1 + 4mr^2/MR^2)^2} \right) = 0$$

This gives

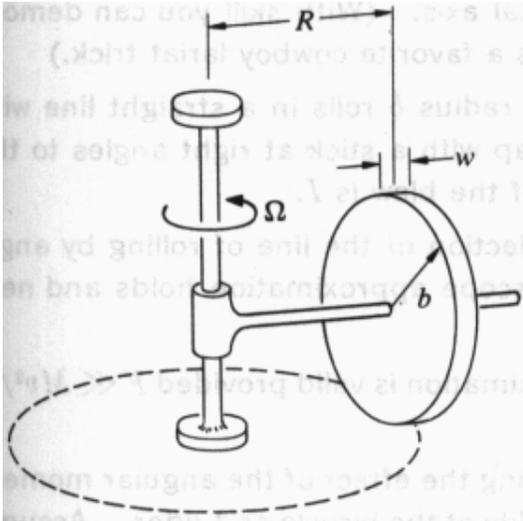
$$1 - \frac{16mr^2}{MR^2} \frac{1}{1 + 4mr^2/MR^2} = 0$$

We can solve this for r . Set $x = 4mr^2/MR^2$. Then $1 - (4x/(1+x)) = 0$ so $x = 1/3$, and

$$\frac{4mr^2}{MR^2} = \frac{1}{3} \Rightarrow r = R\sqrt{\frac{M}{12m}}$$

If we plug in the mass values for this problem, we obtain $r = R/\sqrt{6}$. Since the centrifugal force is everywhere positive, and it is zero at the center, this extremum must in fact be the maximum.

5. K&K problem 7.4



Referring to the diagram, the stone orbits around the vertical shaft with orbiting angular velocity Ω . The velocity of the stone's CM is thus $v = \Omega R$. The stone is rolling without slipping on the flat surface, so its rolling angular velocity is

$$\omega = \frac{v}{b} = \Omega \frac{R}{b}$$

in magnitude. Since angular velocity is a vector, we can add these separate components to obtain the full angular velocity vector. In cylindrical coordinates, with $\hat{\mathbf{z}}$ pointing along the axis of the orbit,

$$\boldsymbol{\omega} = -\frac{R}{b}\Omega\hat{\mathbf{r}} + \Omega\hat{\mathbf{z}}$$

where the leading minus sign tells us that the radial component of $\boldsymbol{\omega}$ is negative, *i.e.* the millstone is rotating clockwise about its horizontal axle. To calculate the angular momentum, we choose as an origin the intersection of the centerline of the vertical shaft and the centerline of the horizontal axle. Both the shaft and the axle are parallel to mirror symmetry axes of the millstone; thus we expect that the component of angular momentum due to $\boldsymbol{\Omega}$ will be parallel to $\boldsymbol{\Omega}$, and the component of angular momentum due to $\boldsymbol{\omega}$ will be parallel to $\boldsymbol{\omega}$. More quantitatively, the component L_z along $\hat{\mathbf{z}}$ is equal to $(I' + MR^2)\Omega$, where $I' = \frac{1}{4}Mb^2$ is the moment of inertia of a disk about a *diameter* and MR^2 is added to I' by use of the parallel axis theorem. Since L_z is constant, no torque is required to maintain it and we don't need to consider it further. To calculate the radial component L_r of the angular momentum, we need $I = Mb^2/2$, the moment of inertia of a disk about its *center*:

$$L_r = -\frac{1}{2}Mb^2\omega$$

where the minus sign again reminds us that the millstone is rolling clockwise about its horizontal axle. Remember that, in cylindrical coordinates, the only unit vector which is constant is $\hat{\mathbf{z}}$; the radial and azimuthal unit vectors depend on θ . Even though the magnitude of L_r is constant, its direction is changing. In a time increment dt , the azimuth θ of the millstone axle with respect to the shaft changes by an angular increment $d\theta = \Omega dt$. This causes \mathbf{L} to change by

$$\begin{aligned} d\mathbf{L} &= \hat{\boldsymbol{\theta}}L_r d\theta \\ &= \hat{\boldsymbol{\theta}}L_r \Omega dt \\ &= -\hat{\boldsymbol{\theta}}\frac{1}{2}Mb^2\omega \Omega dt \end{aligned}$$

The torque is thus

$$\tau = \frac{d\mathbf{L}}{dt} = -\hat{\theta} \frac{1}{2} M b^2 \omega \Omega$$

Finally we consider the forces on the system. The vertical shaft exerts a force on the horizontal axle, gravity pulls down on the millstone's CM, and the normal force pushes up on the millstone. However, with respect to the chosen origin, the first of these forces can exert no torque because it is applied at $\mathbf{r} = 0$. The torque due to gravity is in the $+\hat{\theta}$ direction, and the torque due to the normal force N of the flat surface on the millstone is in the $-\hat{\theta}$ direction. Thus, in the $+\hat{\theta}$ direction, we have

$$-NR + MgR = \tau = -\frac{1}{2} M b^2 \omega \Omega$$

$$M \left(g + \frac{1}{2} \frac{b^2}{R} \omega \Omega \right) = N$$

Substituting $\omega = R\Omega/b$,

$$N = M \left(g + \frac{1}{2} b \Omega^2 \right)$$

Of course, by Newton's third law, the contact force exerted by the millstone upon the flat surface is equal and opposite to N . As advertised, the effective weight of the millstone for crushing grain is greater than Mg ; this increment rises quadratically with the angular velocity.

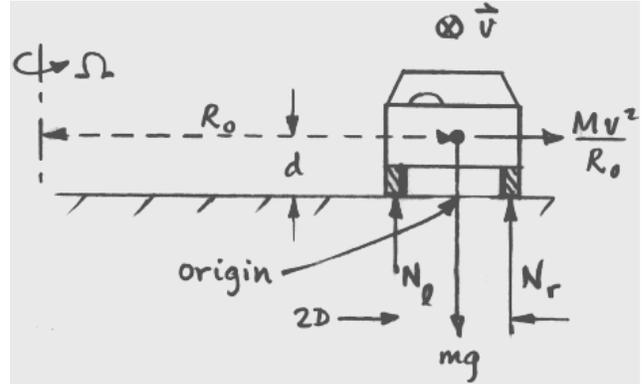
What keeps the millstone from accelerating upward, since the upward normal force on it is greater than the downward force of gravity? The force of the vertical shaft on the horizontal axle, which we ignored in the torque equation because it is applied at the origin, must push downward, in alignment with gravity, with the value $Mb\Omega^2/2$.

Such millstones must have been in use before the time of Newton, so the benefits of their increased effective weight when rolling in a circle must have been discovered empirically rather than logically.

6. K&K problem 7.5

(a.) If the flywheel were horizontal with its spin axis pointing up, it would have little effect, since

the direction of its angular momentum would not change as the car turns left or right. So the flywheel should be vertical with its spin axis pointing either sideways or forward. Deciding between these alternatives requires a more quantitative analysis.



Let's look at the car from the rear while it is in motion with speed v . First we'll consider the car without any flywheel. Suppose that the car is in the process of turning to the left, taking a turn of radius R_0 . Adopt a reference frame attached to the car, with an origin halfway between the tires at the level of the road. It is easy to see why the act of turning causes the normal forces on the tires to become unbalanced. The sum of the torques on the car must remain zero if the car (assumed to have no suspension system, so it doesn't lean) keeps all four tires on the road. With respect to the origin chosen, the forces of friction on both tires can exert no torque, because these forces act directly toward or away from the origin. Neither can the force of gravity exert a torque about this origin, for the same reason. That leaves N_l and N_r , the normal forces on the left and right sets of tires, and Mv^2/R_0 , the fictitious centrifugal force which pulls the CM to the right in this accelerating frame. Let the CM be a distance d above the road; let the right-left separation of the wheels be $2D$. Along $-\hat{v}$, the sum of the torques is then

$$-N_l D + N_r D - \frac{Mv^2}{R_0} d = 0$$

Clearly N_r must exceed N_l if this equation is to be satisfied. This is the problem we are trying to solve with the flywheel.

The torques that we just considered were along $-\hat{v}$. If the flywheel is to help, its angular momentum \mathbf{L} should be directed so that, when the car turns left, the flywheel produces a torque on the car equal to $+Mv^2d/R_0$ along $-\hat{v}$. By Newton's third law, the torque of the car on the flywheel should correspondingly be equal to $+\hat{v}Mv^2d/R_0$. So, as the car turns left, the change in \mathbf{L} of the flywheel should be directed along $+\hat{v}$. This will happen if the angular momentum vector of the (vertical) flywheel is pointing to the right. This means that the flywheel should rotate in the opposite direction as the tires. For simplicity, we'll install it at height d from the road so as not to perturb the CM.

It's not necessary to reverse the flywheel direction for right as opposed to left turns, because both the centrifugal force and the change in \mathbf{L} will correspondingly reverse direction.

(b.) Now that we have determined the flywheel direction, we can calculate the desired magnitude L of the flywheel's angular momentum. We have

$$\frac{dL}{dt} = L\Omega = Mv^2d/R_0$$

where $\Omega = v/R_0$ is the angular velocity of the car around the turn. Solving,

$$L = Mvd$$

For a disk-shaped flywheel of mass m and radius r , $I = mr^2/2$, and the flywheel's angular velocity should be

$$\omega = \frac{2Mvd}{mr^2}$$

This is independent of the turn radius R_0 , which is very nice. We've achieved perfectly flat cornering for a turn of any radius! Unfortunately, ω depends linearly on the velocity v of the car. So, unless we can come up with a quick easy way of varying the kinetic energy of a big flywheel in concert with the square of the car's speed, we're not going to get rich installing these devices as high-performance vehicle options.

7. K&K problem 8.2

(a.) The acceleration of the truck is A , and the mass and width of its rear door are M and w . The door starts fully open. The door can be thought of as a series of thin sticks, pivoted about their ends. The moment of inertia of the door is thus

$$I = \frac{1}{3}Mw^2$$

The easiest way to find the angular velocity of the door is to use work and energy. The rotational kinetic energy is given by

$$K = \frac{1}{2}I\omega^2$$

The work done by a torque on a system is given by

$$W = \int \tau \cdot d\theta$$

In this system there is one torque of interest. We use the hinge of the door as the origin, so the only torque comes from the fictitious force of acceleration. When the door has swung through an angle θ , this torque is given by

$$\tau = \frac{1}{2}MAw \cos \theta$$

Note that $w/2$ is just the distance to the center of mass. From this we can easily calculate the work done from 0 to 90 degrees.

$$W = \int_0^{\pi/2} \frac{1}{2}MAw \cos \theta d\theta = \frac{1}{2}MAw$$

Substituting the expression for rotational kinetic energy, we find the angular velocity of the door after it has swung through 90°:

$$\frac{1}{2}I\omega^2 = \frac{1}{2}MAw \Rightarrow \omega = \sqrt{\frac{3A}{w}}$$

(b.) The force on the door needs to do two things. It needs to accelerate the door at a rate A , and it needs to provide the centripetal acceleration to make the door rotate. At $\theta=90$ degrees, the torque is zero, so the angular velocity is not changing. Instantaneously, the door is in uniform circular motion. The force required to accelerate the door is just

$$F_A = MA$$

The force required to provide the centripetal acceleration is

$$F_c = M\omega^2 \frac{w}{2}$$

The total force is the sum of these, and they act in the same direction. We substitute the value for ω from part (a.) to get

$$F = F_A + F_c = MA + \frac{3}{2}MA = \frac{5}{2}MA$$

8. K&K problem 8.4

(a.) This is a torque balance problem. A car of mass m has front and rear wheels separated by a distance l , and its center of mass is midway between the wheels a distance d off the ground. If the car accelerates at a rate A , it feels a fictitious force acting on the center of mass. This tends to lift the front wheels. When the front wheels are about to lift off the ground, the normal force on the front wheels, N_f is zero. This means that the normal force on the back wheels N_b must equal the weight of the car, $N_b = mg$. The simplest origin to use in this problem is the point on the road directly under the center of mass. Here there are only three torques, due to the two normal forces and fictitious force. The torque from N_b exactly balances the torque from the fictitious force when the wheels are about to lift, so we have

$$\frac{1}{2}N_b l = mA d = \frac{1}{2}mgl \Rightarrow A = \frac{l}{2d}g$$

For the numbers given, $A = 2g = 19.6 \text{ m/sec}^2$.

(b.) For deceleration at a rate g , again we simply apply torque balance about the same origin. We also need the fact that

$$N_f + N_b = mg$$

Torque balance gives

$$\frac{1}{2}N_f l - \frac{1}{2}N_b l - mgd = 0$$

Substituting from the previous equation, we get

$$N_f = \left(\frac{1}{2} + \frac{d}{l}\right) mg$$

$$N_b = \left(\frac{1}{2} - \frac{d}{l}\right) mg$$

Plugging in the numbers, we get $N_f = 3mg/4 = 2400 \text{ lb}$, and $N_b = mg/4 = 800 \text{ lb}$.